TM 5 Pr 8.5

TWO PARTICLES ORBIT EACH OTHER IN CIRCLES WITH A PERIOD OF τ. IF THEY ARE STOPPED AND ALLOWED TO FALL TOWARD EACH OTHER, SHOW THEY WILL COLLIDE IN A TIME

$$t_{COLLIDE} = \frac{\tau}{4\sqrt{2}}$$

APPLY NSL TO ONE OF THE MASSES:

$$\sum F = ma$$

$$-\frac{GM\cancel{m}}{r^2} = \cancel{m} v \frac{dv}{dr}$$

SEPARATING VARIABLES

$$\int_0^v v\,dv = \int_{r_0}^r -\frac{GM}{r^2}\,dr$$

$$\frac{1}{2}v^2 = GM\left(\frac{1}{r}\right|_{r_0}^{r} = GM\left(\frac{1}{r} - \frac{1}{r_0}\right)$$

SOLVING FOR v

$$v = \frac{dr}{dt} = \sqrt{2GM}\sqrt{\frac{1}{r} - \frac{1}{r_0}}$$

SEPARATING VARIABLES

$$\int_0^t dt = \frac{1}{\sqrt{2GM}}\int_{r_0}^{r} \frac{dr}{\sqrt{\frac{1}{r} - \frac{1}{r_0}}}$$

FOLLOWING THE SOLUTION TO PROBLEM TM5 5.5 IN HW 18, USE A TRIG SUBSTITUTION,

$$r = r_0 \cos^2\theta$$

EVALUATE LIMITS: $r = r_0 \Rightarrow \cos^2\theta = 1 \Rightarrow \theta = 0$

$r = 0 \Rightarrow \cos^2\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

THEN

$$t = \frac{2r_0^{3/2}}{\sqrt{2GM}}\int_0^{\theta} \cos^2\theta\,d\theta$$

$\longrightarrow$

INTEGRATING,

$$t = \frac{2r_0^{3/2}}{\sqrt{2GM}}\left[\frac{\theta}{\cancel{2}} + \frac{\sin 2\theta}{4/2}\right]_0^{\theta}$$

$$t = \frac{r_0^{3/2}}{\sqrt{2GM}}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]$$

WHEN $r \to 0$, $\theta \to \frac{\pi}{2}$

$$t_{collide} = \frac{r_0^{3/2}}{\sqrt{2GM}}\left[\frac{\pi}{2} + \frac{1}{2}\underbrace{\sin(\pi)}_{0}\right]$$

$$\Rightarrow \quad t_{collide} = \frac{\pi r_0^{3/2}}{2\sqrt{2GM}}$$

TO GET THIS IN ITS FINAL FORM, FIND AN EXPRESSION FOR T IN TERMS OF r_0.

IN ORBIT: $\quad v = \frac{2\pi r_0}{T} \quad \Rightarrow \quad T = \frac{2\pi r_0}{v}$

AND $\quad \frac{GM\cancel{m}}{r_0^2} = \cancel{m}\frac{v^2}{\cancel{r_0}} \quad \Rightarrow \quad v = \sqrt{\frac{GM}{r_0}}$

COMBINING THESE,

$$T = 2\pi r_0\sqrt{\frac{r_0}{GM}} = \frac{2\pi r_0^{3/2}}{\sqrt{GM}}$$

SUBSTITUTING IN $t_{collide}$ GIVES

$$\boxed{t_{collide} = \frac{T}{4\sqrt{2}}} \quad \text{QED!}$$

NOTE BOOK

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